THE SPECIALIZATION INDEX OF A VARIETY OVER A DISCRETELY VALUED FIELD

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ABSTRACT. Let X be a proper variety over a henselian discretely valued field. An important obstruction to the existence of a rational point on X is the *index*, the minimal positive degree of a zero-cycle on X. This paper introduces a new invariant, the *specialization index*, which is a closer approximation of the existence of a rational point. We provide an explicit formula for the specialization index in terms of an *snc*-model, and we give examples of curves where the index equals one but the specialization index is different from one, and thus explains the absence of a rational point. Our main result states that the specialization index of a smooth, proper, geometrically connected $\mathbb{C}((t))$-variety with trivial coherent cohomology is equal to one.

1. INTRODUCTION

Let R be a henselian discrete valuation ring with quotient field K and algebraically closed residue field k. The C_1 conjecture predicts that every separably rationally connected K-variety has a rational point[1]. This is known when R has equal characteristic: by Greenberg approximation one can assume that the variety is defined over a function field, in which case the statement was proven by Grabber-Harris-Starr in characteristic zero [GHS03] and de Jong-Starr in positive characteristic [dJS03] by means of geometric deformation arguments. However, no direct proof (avoiding Greenberg approximation) is known, and the mixed characteristic case is still wide open. For these purposes, it is desirable to obtain a motivic understanding of the result in equal characteristic, which would have better chances to generalize to mixed characteristic. A fundamental result in this direction is Esnault's proof of the existence of a rational point on rationally chain connected varieties over finite fields [Esn03].

An important obstruction to the existence of a rational point on a K-scheme of finite type X is given by the *index* $\iota(X)$. It is the greatest common divisor of the degrees of the closed points on X, or, equivalently, the minimum of the set of integers $d > 0$ such that X has a zero-cycle of degree d. If X has a K-rational point, then $\iota(X)$ equals one, but the converse does not hold in general. In many interesting situations, the index can be computed in a motivic or cohomological way. For instance, Esnault, Levine and Wittenberg have proven that, if X is proper over K and we denote by $p \geq 1$ the characteristic exponent of k, then the prime-to-p part of the index $\iota(X)$ divides the Euler characteristic of any coherent sheaf on X [ELW12, 3.1]. It follows that, when k has characteristic zero, every rationally connected K-variety X has index one, since the higher coherent cohomology spaces of X vanish

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[1]To be precise, the field K is not necessarily C_1; but on smooth K-varieties, the existence of a rational point is equivalent to the existence of a point over the completion of K, which is a C_1 field.

so that the coherent Euler characteristic of X equals one. The latter result has been proven independently by several authors; we refer to the introduction of [ELW12] for precise references. Beware, however, that the triviality of the coherent cohomology is not sufficient to guarantee the existence of a rational point: in [Laf04], Lafon has constructed an example of an Enriques surface over $\mathbb{C}((t))$ without rational points.

The purpose of the present paper is to introduce a finer invariant than the index, which we call the *specialization index*, for proper K-schemes of finite type. It is always bounded by the index and it equals one if X has a rational point. The interest of this invariant lies in the fact that there exist many proper K-schemes of index one whose specialization index is strictly larger than one. In such examples, the specialization index explains the non-existence of a rational point, whereas the index does not. Our main result (Theorem 4.6) states that, when k has characteristic zero and X is a smooth, proper, geometrically connected K-variety with trivial coherent cohomology, then the specialization index of X equals one. This strengthens the aforementioned result for the index. In a sense, this result is disappointing, because it means that the specialization index does not allow to distinguish between varieties with and without rational points in the case of trivial coherent cohomology. Our proof is inspired by, and makes use of, the Woods Hole trace formula [SGA5, III.6.12], which implies that over a *finite* field, every smooth and proper variety with trivial coherent cohomology has a rational point.

Let us give a brief overview of the structure of the paper. In Section 2, we will define the specialization index and give an explicit way to compute it in terms of the special fiber of an *snc*-model. We then use this formula, together with Winters's classification of combinatorial reduction types of curves, to produce examples of curves C with $\iota(C) = 1$ and $\iota_{\mathrm{sp}}(C) > 1$ in Section 3. In these examples, the specialization index explains the absence of a rational point, while the index does not. In Section 4, we prove our main result, Theorem 4.6. The proof is based on Hodge theory, the Woods Hole trace formula and a description of the behavior of *snc*-models under ramified base change.

Notation and terminology.

(1.1) Let R be a henselian discrete valuation ring with algebraically closed residue field k and quotient field K. We fix an algebraic closure K^a of K. The integral closure R^a of R in K^a is a valuation ring with quotient field K^a and residue field k (see [Neu99, II.6.2]). We denote by p the characteristic exponent of k.

(1.2) Let L be a finite extension of K. Since K is henselian, the discrete valuation on K extends uniquely to a valuation on L, whose valuation ring is the integral closure of R in L [Neu99, II.6.2]. If we denote by e the ramification index of L over K, then Ostrowski's Lemma implies that $[L : K] = \delta e$ with δ a power of p (see Theorem 2 in [Rib99, 6.1]). The invariant δ is called the *defect* of the extension L/K. We say that the extension L/K is defectless if $\delta = 1$; this is always the case

if L is separable over K or R is excellent (in particular, if R is complete), as can be deduced from Theorem 2 and Corollary 1 in [Bou72, VI.8.5].

(1.3) For every R-scheme $\mathcal{X}$, we denote by $\mathcal{X}_K = \mathcal{X} \times_R K$ its generic fiber and by $\mathcal{X}_k = \mathcal{X} \times_R k$ its special fiber. If $\mathcal{X}$ is proper over R, we can consider the specialization map

$$\mathrm{sp}_{\mathcal{X}} : \mathcal{X}_K(K^a) \to \mathcal{X}_k(k).$$

It is defined by composing the inverse of the bijection $\mathcal{X}(R^a) \to \mathcal{X}_K(K^a)$ with the reduction map $\mathcal{X}(R^a) \to \mathcal{X}_k(k)$.

(1.4) Let X be a proper K-scheme. A *model* for X is a proper flat R-scheme $\mathcal{X}$ endowed with an isomorphism of K-schemes $\mathcal{X}_K \to X$. Nagata's compactification theorem implies that a model of X always exists. A morphism of models is defined in the obvious way. If $\mathcal{X}$ and $\mathcal{X}'$ are models of X then we say that $\mathcal{X}'$ dominates $\mathcal{X}$ if there exists a morphism of models $\mathcal{X}' \to \mathcal{X}$ (such a morphism is unique if it exists). The domination relation defines a partial ordering $\geq$ on the set of isomorphism classes of R-models of X.

(1.5) We say that $\mathcal{X}$ is an *snc-model* of X if $\mathcal{X}$ is regular and $\mathcal{X}_k$ is a divisor with strict normal crossings. If $p = 1$, then every regular proper K-scheme X has an *snc*-model by Hironaka's resolution of singularities. More precisely, for every model $\mathcal{X}$ of X there exists a morphism of models $\mathcal{X}' \to \mathcal{X}$ such that $\mathcal{X}'$ is an *snc*-model of X[2]. By [CP14, 1.1] and [CJS09, 0.3], this also holds for $p \geq 2$ provided that R is excellent and the dimension of X is at most two. If $p = 1$ or X is a curve, one can arrange that $\mathcal{X}' \to \mathcal{X}$ is an isomorphism over the *snc*-locus of $\mathcal{X}$, but this does not seem to follow directly from the statements in [CJS09] and [CP14].

(1.6) For every regular proper K-scheme X and every *snc*-model $\mathcal{X}$ of X, we introduce the following notation. We write

$$\mathcal{X}_k = \sum_{i \in I} N_i E_i$$

with E_i, $i \in I$ the irreducible components of $\mathcal{X}_k$ and N_i the multiplicity of E_i in the divisor $\mathcal{X}_k$. For every non-empty subset J of I we set

$$E_J = \bigcap_{j \in J} E_i, \quad E_J^o = E_J \setminus \left(\bigcup_{i \notin J} E_i \right).$$

The sets E_J form a partition of $\mathcal{X}_k$ into locally closed subsets. We also write

$$N_J = \gcd\{N_j \,|\, j \in J\}.$$

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[2]More precisely, Hironaka showed that we can find such a morphism as a composition of blow-ups at centers in the special fiber if R is complete. The general case then follows from the fact that blowing up commute with flat base change and that the *snc* property can be checked after base change to the completion.

2. THE SPECIALIZATION INDEX

Definition 2.1 (The specialization index)**.** For every proper R-scheme $\mathcal{X}$, we denote by $\mathcal{D}_\mathcal{X}$ the set of integers $d > 0$ such that there exists a zero-cycle of degree d on $\mathcal{X}_K$ whose support is contained in $\mathrm{sp}_\mathcal{X}^{-1}(x)$ for some $x \in \mathcal{X}(k)$. We call the minimum of $\mathcal{D}_\mathcal{X}$ the specialization index of $\mathcal{X}$, and we denote it by $\iota_{\mathrm{sp}}(\mathcal{X})$.

For every proper K-scheme X, we set

$$\mathcal{D}_X = \bigcap \{ \mathcal{D}_\mathcal{X} \, | \, \mathcal{X} \text{ is an } R\text{-model of } X \}.$$

We call the minimum of $\mathcal{D}_X$ the specialization index of X, and we denote it by $\iota_{\mathrm{sp}}(X)$.

(2.2) Note that $\mathcal{D}_X$ is non-empty, because for every closed point x of X, the degree of x belongs to $\mathcal{D}_X$. It is obvious from the definition that $\iota_{\mathrm{sp}}(X) \geq \iota_{\mathrm{sp}}(\mathcal{X})$ for every model $\mathcal{X}$ of X, and that $\mathcal{D}_{\mathcal{X}'} \subseteq \mathcal{D}_\mathcal{X}$ (and hence $\iota_{\mathrm{sp}}(\mathcal{X}') \geq \iota_{\mathrm{sp}}(\mathcal{X})$) if $\mathcal{X}$ and $\mathcal{X}'$ are models of X such that $\mathcal{X}'$ dominates $\mathcal{X}$. If we denote by $\iota(X)$ the index of X and by $\nu(X)$ its ν-invariant (that is, the minimal degree of a closed point of X), then we obviously have $\iota(X) | \iota_{\mathrm{sp}}(X)$ and $\iota_{\mathrm{sp}}(X) \leq \nu(X)$. In particular, if X has a rational point, then $\iota(X) = \iota_{\mathrm{sp}}(X) = 1$.

(2.3) At first sight, it seems difficult to compute the specialization index $\iota_{\mathrm{sp}}(X)$ of a proper K-scheme X, since the definition suggests that we have to check all possible models of X. We will show in Proposition 2.9 that, if every model of X can be dominated by an *snc*-model, the specialization index can be computed on a single *snc*-model. First, we need to establish some auxiliary results.

Lemma 2.4. *Let X be a proper K-variety and let $\mathcal{X}$ be a model of X. Suppose that $y \in \mathcal{X}_k$ is a closed point such that both $\mathcal{X}$ and $(\mathcal{X}_k)_{\mathrm{red}}$ are regular at y, and denote by N the multiplicity of $\mathcal{X}_k$ at y. Then there is a closed point $x \in X$ that specializes to y and such that residue field at x is a defectless extension of K of degree N.*

Proof. The following proof is taken from [Wit15, 4.6]. Let π be a uniformizer in R. Then we can find elements u and t in $\mathcal{O}_{\mathcal{X},y}$ such that u is a unit, $t = 0$ is a local equation for $(\mathcal{X}_k)_{\mathrm{red}}$ at y, and $\pi = ut^N$. We can extend t to a regular system of local parameters $(t, f_1, \ldots, f_n)$ in $\mathcal{O}_{\mathcal{X},y}$. We set

$$R' = \frac{\mathcal{O}_{\mathcal{X},y}}{(f_1, \ldots, f_n)}.$$

Then R' is a regular local ring of dimension 1, and thus a discrete valuation ring. The residue class of t is a uniformizer in R' and its residue field $R'/(t)$ is isomorphic to k, the residue field of $\mathcal{X}$ at y. The natural morphism $R \to R'$ is flat because it maps π to the non-zero element ut^N.

Take $U = \mathrm{Spec}\, A$ an affine open around y such that $f_1, \ldots, f_n$ and t are defined on U. If we take

$$V = \mathrm{Spec}\, \frac{A}{(f_1, \ldots, f_n)},$$

then $y \in V$ and $\mathcal{O}_{V,y} = R'$. Now $f : V \to \mathrm{Spec}\, R$ is of finite type and the image of y is the closed point of $\mathrm{Spec}\, R$. Thus, [EGA4, 18.5.11(c')] implies that R' is a finite R-algebra. Since R' is flat over R and $R'/\pi R' \cong R'/(t^N)$, we know that it is free of

rank N. Hence, the fraction field K' of R' is an extension of K of degree N. If we denote by x the image of the generic point of $\operatorname{Spec} R'$ under the natural morphism

$$\operatorname{Spec} R' \to \mathcal{X},$$

then x is a closed point of degree N on X and $\operatorname{sp}_{\mathcal{X}}(x) = y$. $\qquad\square$

Proposition 2.5. *Let X be a proper K-scheme and let $\mathcal{X}$ be a model for X. Let y be a closed point of $\mathcal{X}_k$ such that $\mathcal{X}$ is regular at y and $\mathcal{X}_k$ has strict normal crossings at y. Denote by E_j, $j \in J$ the irreducible components of $\mathcal{X}_k$ that pass through y and by N_j the multiplicity of $\mathcal{X}_k$ along E_j.*

Then for every positive integer d, the following properties are equivalent:

(1) there exists a closed point x on X whose degree divides d and such that $\operatorname{sp}_{\mathcal{X}}(x) = y$;

(2) there exists a tuple α in $(\mathbb{Z}_{>0})^J$ such that

$$\sum_{j \in J} \alpha_j N_j = d.$$

Proof. Assume that (1) holds. We choose a uniformizer π of R. Since $\mathcal{X}$ is regular at y, we can choose a local equation $z_j = 0$ for E_j at y, for every $j \in J$, and write

$$\pi = u \prod_{j \in J} z_j^{N_j}$$

in $\mathcal{O}_{\mathcal{X}, y}$, with u a unit. If we denote by v the normalized discrete valuation on the residue field at x, we find

$$(2.6) \qquad\qquad v(\pi) = \sum_{j \in J} v(z_j(x)) N_j.$$

But $v(\pi)$ divides the degree of x, and therefore d. Moreover, $v(z_j(x)) > 0$ for every j in J since x specializes to a point in the zero locus of z_j. Thus, (2) follows by multiplying (2.6) with $d/v(\pi)$.

Now suppose, conversely, that (2) holds. By Lemma 2.4, it suffices to find a morphism $h : \mathcal{X}' \to \mathcal{X}$ of models of X and a closed point y' on $\mathcal{X}'_k$ such that $h(y') = y$, $\mathcal{X}'$ and $(\mathcal{X}')_{\mathrm{red}}$ are regular at y', and the multiplicity of $\mathcal{X}'_k$ at y' divides d. We will construct h by means of a toroidal blow-up. The most convenient way to express the construction is the language of log geometry; we will in particular make use of the notion of log regularity developed in [Kat94]. For every flat R-scheme of finite type $\mathcal{Y}$, we denote by $\mathcal{Y}^+$ the log scheme that we obtain by endowing $\mathcal{Y}$ with the divisorial log structure associated with $\mathcal{Y}_k$. Then $\mathcal{X}^+$ is log regular at y and the characteristic monoid $\overline{\mathcal{M}}_{\mathcal{X}^+, y}$ is canonically isomorphic to $\mathbb{N}^J$, because we assumed that $\mathcal{X}_k$ is a strict normal crossings divisor at y. The residue class of π in $\overline{\mathcal{M}}_{\mathcal{X}^+, y}$ is the multiplicity vector $m = (N_j)_{j \in J}$. We subdivide the cone $\mathbb{R}^J_{\geq 0}$ into a fan Σ by adding the ray generated by $\alpha = (\alpha_j)_{j \in J}$. By [Kat94, 10.3], this subdivision gives rise to a proper birational morphism $h : \mathcal{X}' \to \mathcal{X}$ that is an isomorphism over the complement of y and such that $(\mathcal{X}')^+$ is log regular (in particular, normal) at every point of $E = h^{-1}(y)$. Moreover, E is an irreducible divisor on $\mathcal{X}'$ and the multiplicity of $\mathcal{X}'_k$ along E is given by the scalar product $m \cdot \alpha'$ where α' is a primitive generator of the ray $\mathbb{R}_{\geq 0} \alpha$. By our assumption that $m \cdot \alpha = d$, we know that $m \cdot \alpha'$ divides d. Thus, we can take for y' any closed point on E such that $\mathcal{X}'$ and $(\mathcal{X}'_k)_{\mathrm{red}}$ are regular at y'. Such a point exists because $\mathcal{X}'$ is normal and hence regular in codimension one. $\qquad\square$

Corollary 2.7. *Let X be a regular proper K-scheme and assume that X admits an snc-model $\mathcal{X}$, with $\mathcal{X}_k = \sum_{i \in I} N_i E_i$. Then*

$$\mathcal{D}_\mathcal{X} = \bigcup_{\emptyset \neq J \subset I, E_J^o \neq \emptyset} N_J \cdot \mathbb{Z}_{>0}.$$

Moreover, we have the following equalities:

$$\begin{aligned}
\nu(X) &= \min\{N_i \,|\, i \in I\}, \\
\iota(X) &= \gcd\{N_i \,|\, i \in I\}, \\
\iota_{\mathrm{sp}}(\mathcal{X}) &= \min\{N_J \,|\, \emptyset \neq J \subset I, \, E_J^o \neq \emptyset\}.
\end{aligned}$$

Proof. This follows at once from Proposition 2.5 (and the formulas for $\nu(X)$ and $\iota(X)$ are well-known). $\qquad\square$

We will now prove that the specialization index can be computed on any *snc*-model. For this, we will need the following combinatorial lemma.

Lemma 2.8. *Let S be a subset of $\mathbb{Z}_{>0}$ and suppose that S is the union of finitely many sub-semigroups $S_1, \ldots, S_t$ of $(\mathbb{N}, +)$. If we set $s_i = \gcd S_i$ for every $i \in \{1, \ldots, t\}$, then the value*

$$s_{\min} = \min\{s_i \,|\, i \in \{1, \ldots, t\}\}$$

is independent of the choice of the S_i.

Proof. Let S' be a sub-semigroup of $(\mathbb{N}, +)$ that is contained in S. We set $s' = \gcd S'$. Then it suffices to show that $s_{\min} \leq s'$. We will prove that, for every sufficiently large positive integer m, the element ms' lies in S', and thus in one of the semigroups S_i. This means that ms' is divisible by one of the s_i for every sufficiently large m, so that s' itself must be divisible by one of the s_i and, in particular, $s_{\min} \leq s'$.

By Bezout's lemma, we can write

$$s' = \sum_{j=1}^{n} \lambda_j x_j$$

with $\lambda_j \in \mathbb{Z}$ and $x_j \in S'$. Reordering terms, we may assume that there exists an integer q such that $1 \leq q \leq n$, $\lambda_j > 0$ for $j \leq q$, and $\lambda_j < 0$ for $j > q$. Now set

$$m_0 = -(x_1 - 1) \sum_{j=q+1}^{n} \lambda_j (x_j/s')$$

and choose any integer $m > m_0$. We consider the Euclidean division $m - m_0 = \alpha(x_1/s') + r$ with α a non-negative integer and r an element in $\{0, \ldots, (x_1/s') - 1\}$. Then we can write

$$\begin{aligned}
ms' &= \alpha x_1 + rs' - (x_1 - 1) \sum_{j=q+1}^{n} \lambda_j x_j \\
&= \alpha x_1 + r \sum_{j=1}^{q} \lambda_j x_j - (x_1 - 1 - r) \sum_{j=q+1}^{n} \lambda_j x_j
\end{aligned}$$

which is a positive linear combination of the elements $x_1, \ldots, x_n$ of the semi-group S'. It follows that ms' lies in S' as soon as $m > m_0$. $\qquad\square$

Proposition 2.9. *Let X be a regular proper K-scheme and assume that every model of X can be dominated by an snc-model. Then for every snc-model $\mathcal{X}$ of X, we have $\iota_{\mathrm{sp}}(X) = \iota_{\mathrm{sp}}(\mathcal{X})$.*

Proof. We have already observed that $\mathcal{D}(\mathcal{X}') \subseteq \mathcal{D}(\mathcal{X})$ if $\mathcal{X}'$ is a model of X that dominates $\mathcal{X}$. Thus we only need to argue that $\iota_{\mathrm{sp}}(\mathcal{X})$ is independent of the choice of the *snc*-model $\mathcal{X}$. We write

$$\mathcal{X}_k = \sum_{i \in I} N_i E_i$$

and we denote by S the subset of $\mathbb{Z}_{>0}$ consisting of all the multiples of the degrees of the closed points on X. For every non-empty subset J of I, we set

$$S_J = \left\{ \sum_{j \in J} \alpha_j N_j \mid \alpha_j \in \mathbb{Z}_{>0} \text{ for all } j \in J \right\} \subset \mathbb{Z}_{>0}.$$

Then $N_J = \gcd S_J$. We denote by $\mathcal{J}$ the set of non-empty subsets J of I such that $E_J^o \neq \emptyset$. If J belongs to $\mathcal{J}$, then by Proposition 2.5, the set S_J consists precisely of the multiples of the degrees of the closed points on X that specialize to a point on E_J^o. Since the sets E_J^o form a partition of $\mathcal{X}_k$, we find that

$$S = \bigcup_{J \in \mathcal{J}} S_J.$$

Now Lemma 2.8 implies that $\min\{N_J \mid J \in \mathcal{J}\}$ is independent of the choice of the *snc*-model $\mathcal{X}$; but this is precisely $\iota_{\mathrm{sp}}(\mathcal{X})$, by Corollary 2.7. $\qquad\square$

Corollary 2.10. *Let X be a regular proper K-scheme and assume that every model of X can be dominated by an snc-model. If $\mathcal{X}$ is an snc-model of X, with*

$$\mathcal{X}_k = \sum_{i \in I} N_i E_i,$$

then the specialization index of X can be computed as

$$\iota_{\mathrm{sp}}(X) = \min\{N_J \mid \emptyset \neq J \subset I, \ E_J^o \neq \emptyset\}.$$

Proof. This is a combination of Proposition 2.9 and Corollary 2.7. $\qquad\square$

We recall that the assumption in the statements of Proposition 2.9 and Corollary 2.10 is satisfied when $p = 1$, or $\dim(X) \leq 2$ and R is excellent (see (1.5)). One can reduce to the excellent case by means of the following result.

Proposition 2.11. *Denote by $\widehat{R}$ the completion of R and by $\widehat{K}$ its quotient field. Let X be a regular proper K-scheme, and assume that every $\widehat{R}$-model of $X \times_K \widehat{K}$ can be dominated by a regular $\widehat{R}$-model. Then $\iota_{\mathrm{sp}}(\mathcal{X}) = \iota_{\mathrm{sp}}(\mathcal{X} \times_R \widehat{R})$ for every R-model $\mathcal{X}$ of X, and $\iota_{\mathrm{sp}}(X) = \iota_{\mathrm{sp}}(X \times_K \widehat{K})$.*

Proof. We first show that the base changes to $\widehat{R}$ of the R-models of X are cofinal in the partially ordered set of isomorphism classes of $\widehat{R}$-models of $X \times_K \widehat{K}$. If we fix any R-model $\mathcal{X}$ of X, then we can dominate every $\widehat{R}$-model of $X \times_K \widehat{K}$ by an admissible blow-up of $\mathcal{X} \times_R \widehat{R}$ (that is, the blow-up of an ideal that contains a power of the maximal ideal of R; this is one of the cornerstones of Raynaud's formal-rigid geometry [BL93, 4.1], and it also holds for proper algebraic schemes by formal GAGA and the fact that admissible blow-ups commute with formal

completion). But every admissible blow-up of $\mathcal{X} \times_R \widehat{R}$ is defined over R, since blowing up commutes with flat base change. This implies that it is enough to prove the first assertion in the statement.

Let $\mathcal{X}$ be an R-model of X. It is obvious that $\iota_{\mathrm{sp}}(\mathcal{X}) \geq \iota_{\mathrm{sp}}(\mathcal{X} \times_R \widehat{R})$. Thus it suffices to show that, for every positive integer d and every closed point $\widehat{x}$ of degree d on $X \times_K \widehat{K}$, there exists a closed point x on X whose degree divides d and such that

$$\mathrm{sp}_{\mathcal{X} \times_R \widehat{R}}(\widehat{x}) = \mathrm{sp}_{\mathcal{X}}(x).$$

Let $\mathcal{Y}$ be a regular $\widehat{R}$-model of X that dominates $\mathcal{X} \times_R \widehat{R}$. By repeatedly blowing up $\mathcal{Y}$ at the specialization of $\widehat{x}$, we can arrange that $\mathrm{sp}_{\mathcal{Y}}(\widehat{x})$ lies on a unique irreducible component of $\mathcal{Y}_k$, whose multiplicity N must divide d (note that the multiplicity of the exceptional divisor will strictly increase as long as the center of the blow-up lies on more than one component; but this multiplicity is also bounded by d). Let $\mathcal{Y}'$ be the blow-up of $\mathcal{Y}$ at $\mathrm{sp}_{\mathcal{Y}}(\widehat{x})$, and denote by E' the exceptional divisor. By the first part of the proof, we can dominate $\mathcal{Y}'$ by a model $\mathcal{Y}''$ that is defined over R (but which is not necessarily regular); we write $\mathcal{Y}'' \cong \mathcal{Z} \times_R \widehat{R}$ for some model $\mathcal{Z}$ of X over R. If we denote by E'' the strict transform of E' in $\mathcal{Y}''$, then we can also view E'' as an irreducible component of $\mathcal{Z}_k$. This component still has multiplicity N, and $\mathcal{Z}$ is regular at the generic point of E'' since $\mathcal{Y}'$ is regular at the generic point of E'. Now it follows from Lemma 2.4 that there exists a point x on X of degree N such that $\mathrm{sp}_{\mathcal{Z}}(x)$ lies on E'', which implies that

$$\mathrm{sp}_{\mathcal{X} \times_R \widehat{R}}(\widehat{x}) = \mathrm{sp}_{\mathcal{X}}(x).$$

$\square$

3. Specialization indices of curves

(3.1) In this section, we will show that the specialization index ι_{sp} is a strictly finer invariant than the index ι. Our examples will all be constructed with the help of Winters's classification of reduction types of curves. Let X be a smooth, proper, geometrically connected K-curve, and let $\mathcal{X}$ be an snc-model of X. We write

$$\mathcal{X}_k = \sum_{i=1}^{r} N_i E_i,$$

we denote by G_i the genus of E_i, for every i, and we set $N = (N_1, \ldots, N_r)$ and $G = (G_1, \ldots, G_r)$. Moreover, we set $c_{ij} = E_i \cdot E_j$ for all i, j. Then the matrix $C = (c_{ij})_{i,j}$ is a symmetric $r \times r$ matrix with integer coefficients, and $C \cdot N^t = 0$ because $E_i \cdot \mathcal{X}_k = 0$ for every i. We say that $\mathcal{X}$ has *reduction type* (N, G, C).

Theorem 3.2 (Winters)**.** *Assume that k has characteristic zero and that R is the henselization of $\mathbb{A}_k^1$ at the origin. Let r be a positive integer, let N be a primitive vector in $\mathbb{Z}_{>0}^r$, let G be an element of $\mathbb{N}^r$ and let $C = (c_{ij})$ be a symmetric $r \times r$ matrix with integer coefficients such that $c_{ij} \geq 0$ for $i \neq j$ and $C \cdot N^t = 0$. Then there exist a smooth, proper, geometrically connected K-curve X and an snc-model $\mathcal{X}$ of X of reduction type (N, G, C). Moreover, the genus $g(X)$ of X can be computed from the formula*

$$2 - 2g(X) = \sum_{i=1}^{r} N_i(2 - 2G_i + c_{ii}).$$

Proof. The existence of X and $\mathcal{X}$ follows from [Win74, 3.7 and 4.3]. The formula for the genus X is an elementary consequence of Riemann-Roch and adjunction; see for instance [Nic13, 3.1]. $\square$

Note that Theorem 3.2 remains valid if k has characteristic zero and R is complete, since we can choose an isomorphism between R and the completion of the local ring of $\mathbb{A}_k^1$ at the origin and perform a base change on the model given by Theorem 3.2.

Example 3.3. Assume that k has characteristic zero and that K is complete. We choose a non-negative integer x. We set $r = 5$ and

$$\begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 3 \\ 4 \\ 6 \end{pmatrix}, \quad \begin{pmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \\ G_5 \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} -3 & 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & -4 & 0 & 2 \\ 0 & 1 & 0 & -2 & 1 \\ 1 & 0 & 2 & 1 & -2 \end{pmatrix}.$$

Let $\mathcal{X}$ be an *snc*-model of reduction type (N, G, C), with generic fiber X. Then we can compute by means of Corollaries 2.7 and 2.10 that $\iota(X) = 1$ whereas $\iota_{\mathrm{sp}}(\mathcal{X}) = 2$. Using the formula in Theorem 3.2, we also see that $g(X) = 5 + 2x$.
A picture of the dual graph of the special fiber can be found in Figure 1.
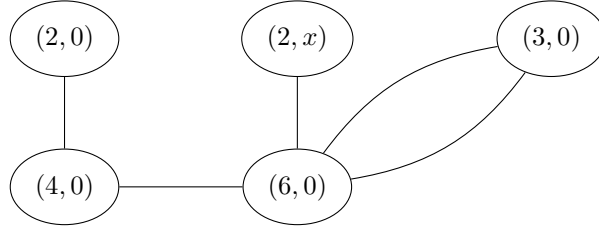

FIGURE 1. The dual graph of the special fiber $\mathcal{X}_k$ of Example 3.3. The vertex corresponding to a component E_i is labelled with (N_i, G_i).

Example 3.4. Assume that k has characteristic zero and that K is complete. We choose a non-negative integer x. We set $r = 6$ and

$$\begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \\ N_6 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 3 \\ 3 \\ 4 \\ 6 \end{pmatrix}, \quad \begin{pmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \\ G_5 \\ G_6 \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} -3 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 & -2 & 1 \\ 1 & 0 & 1 & 1 & 1 & -2 \end{pmatrix}.$$

Let $\mathcal{X}$ be an *snc*-model of reduction type (N, G, C), with generic fiber X. Then we again find that $\iota(X) = 1$ and $\iota_{\mathrm{sp}}(\mathcal{X}) = 2$. In this example, $g(X) = 2 + 2x$.
A picture of dual graph of the special fiber can be found in Figure 2.

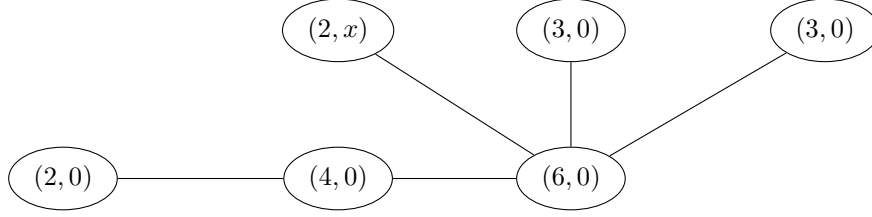Combining Examples 3.3 and 3.4, we arrive at the following proposition.

FIGURE 2. The dual graph of the special fiber $\mathcal{X}_k$ of Example 3.4. The vertex corresponding to a component E_i is labelled with (N_i, G_i).

Proposition 3.5. *Assume that k has characteristic zero and that K is complete. If $g = 2$ or $g \geq 4$, then there exists a smooth, proper, geometrically connected K-curve C of genus g such that $\iota(C) = 1$ while $\iota_{\mathrm{sp}}(C) > 1$.*

Proof. This follows from Examples 3.3 and 3.4, since we can write g as $2 + 2x$ or $5 + 2x$ with x a non-negative integer. $\qquad\square$

(3.6) Proposition 3.5 does not cover the cases $g \leq 1$ and $g = 3$. As a matter of fact, we will now explain that in those cases, the specialization index is one if and only if the index is one. Let X be a smooth, proper and geometrically connected K-curve. If X has genus zero then it has a rational point by the triviality of the Brauer group of K, so that $\iota(X) = \iota_{\mathrm{sp}}(X) = 1$. If X has genus one and D is a divisor of degree $\iota(X)$ on X, then it follows from Riemann-Roch that D is linearly equivalent to an effective divisor D' on X. Then any closed point in the support of D' has degree at most $\iota(X)$, so that $\iota(X) = \iota_{\mathrm{sp}}(X) = \nu(X)$. This value is also equal to the period of X (the order of the class of X in the Weil-Châtelet group of its Jacobian) by the triviality of the Brauer group of K [Lic68, Theorem 1]. The genus 3 case is treated in the following proposition.

Proposition 3.7. *Let X be a smooth, proper, geometrically connected K-curve of genus 3. If $\iota(X) = 1$, then $\iota_{\mathrm{sp}}(X) = 1$.*

Proof. We have seen in Corollaries 2.7 and 2.10 that $\iota(X)$ and $\iota_{\mathrm{sp}}(X)$ only depend on the dual graph of the special fiber of an *snc*-model $\mathcal{X}$ of X, where we label each vertex with the multiplicity of the corresponding component. The assumption that $\iota(X) = 1$ is equivalent to the property that the multiplicity vector N of $\mathcal{X}$ is primitive, by Corollary 2.7. Denote by K' the fraction field of the henselization of $\mathbb{A}_k^1$ at the origin. By Winters's theorem (Theorem 3.2), we can always find a smooth, proper and geometrically connected curve X' of genus 3 over K' with an *snc*-model whose special fiber has the same dual graph (with the same labels) as $\mathcal{X}_k$. Thus we may assume that $K = K'$. Then we can deduce the statement from the classification of reduction types of minimal regular models of genus 3 curves over complex discs in [Uem99] (alternatively, one can also check that the results we need from that paper remain valid for arbitrary R, with the same arguments).

We must show that there is no case where $\iota(X) = 1$ but $\iota_{\mathrm{sp}}(X) > 1$. This means that we only need to look at minimal regular models $\mathcal{X}$ whose special fiber does not contain an irreducible component of multiplicity one, or two irreducible components of coprime multiplicities that intersect transversally in a point (in these

cases, $\iota_{\mathrm{sp}}(X) = 1$ by Corollary 2.10 and the fact that we can modify $\mathcal{X}$ into an *snc*-model without changing its *snc*-locus).

The cases left to consider are the following combinations of trunks: [2G], [2H], [4I], [4J], [2I,2I], [2I,2J], [2J,2J] (see [Uem99, Figure 3]). In order to have $\iota(X) = 1$, the special fiber of the minimal regular model $\mathcal{X}$ must contain a branch of type (b) with $p = 6$ or (h) with $p = 6$ and $q = 4$ (see [Uem99, Figure 2]). All the components of the other branches have multiplicities divisible by 2. This is only possible with the trunk combinations [2H] and [2J,2J]. In both cases, we would then need an additional branch of type (b) with $p = 2$, but these branches have a component with multiplicity one, so that $\iota_{\mathrm{sp}}(X) = 1$. □

(3.8) To conclude this section, we will establish an upper bound for the specialization index of a curve in terms of its genus. Let X be a smooth, proper, geometrically connected K-curve of genus g. We have already observed in (3.6) that X has a rational point (thus $\iota(X) = \iota_{\mathrm{sp}}(X) = 1$) if $g = 0$, and that $\iota(X)$ and $\iota_{\mathrm{sp}}(X)$ are both equal to the period of X if $g = 1$. This period can be arbitrarily large: if E is an elliptic K-curve with semi-stable reduction, then $H^1(K, E)$ contains elements of arbitrarily large order, by [Sha61]. However, if $g \geq 2$, we obtain an upper bound on $\iota(X)$ from Riemann-Roch: the degree of the canonical divisor of X equals $2g - 2$, so that $\iota(X) \leq 2g - 2$. With some extra work, one can bring this bound down to $g - 1$; this result is well-known and can be viewed as a special case of [ELW12, 2.1] since $1 - g$ is the coherent Euler characteristic of C. We observe that the bound $2g - 2$ still holds for the specialization index, as a consequence of the following stronger result.

Proposition 3.9. *Let F be any field and let X be a smooth, proper, geometrically connected F-curve of genus $g \geq 2$. Let m be the smallest positive multiple of the index $\iota(X)$ such that $m \geq g$. Then $m \leq 2g - 2$ and X has a closed point of degree $d \leq m$.*

Proof. Every canonical divisor on X has degree $2g - 2 \geq g$ by Riemann-Roch, so that $m \leq 2g - 2$. Applying Riemann-Roch to any divisor D of degree m on X, we find that D is linearly equivalent to an effective divisor D', and each point in the support of D' has degree at most m. □

(3.10) We do not know if the bound $2g - 2$ on the specialization index is sharp for each fixed genus $g \geq 2$. Unfortunately, we cannot use Winters's result (Theorem 3.2) to produce examples: the theorem assumes that the multiplicity vector N is primitive, in which case the index $\iota(X)$ equals one and the specialization index is at most g, by Proposition 3.9.

4. The specialization index for varieties with trivial coherent cohomology

(4.1) Throughout this section, we assume that k has characteristic zero (that is, $p = 1$) and we denote by X a geometrically connected, proper and smooth K-variety. Our aim is to prove that $\iota_{\mathrm{sp}}(X) = 1$ if X has trivial coherent cohomology. First, we introduce some useful notation. For every positive integer d, we denote by $K(d)$ the unique extension of degree d of K inside the fixed algebraic closure

K^a, and we write $R(d)$ for its valuation ring. We also set $X(d) = X \times_K K(d)$. The extension $K(d)$ is Galois over K, with Galois group $\mu_d(k)$. We choose a geometric monodromy operator, that is, a topological generator σ of the procyclic Galois group $\mathrm{Gal}(K^a/K)$.

Definition 4.2 (Monodromic model)**.** Let d be a positive integer. A *monodromic model* for X over $R(d)$ is a regular model $\mathcal{Y}$ for $X(d)$ over $R(d)$ such that the Galois action of $\mu_d(k)$ on $X(d)$ extends to $\mathcal{Y}$. We say that a monodromic model $\mathcal{Y}$ is *strictly semi-stable* if $\mathcal{Y}_k$ is a reduced divisor with strict normal crossings on $\mathcal{Y}$.

Proposition 4.3. *For every sufficiently divisible positive integer d, there exists a strictly semi-stable monodromic model $\mathcal{Y}$ of X over $R(d)$ such that $\mathcal{Y}$ admits a $\mu_d(k)$-equivariant morphism*

$$\mathcal{Y} \to \mathcal{X} \times_R R(d)$$

of models of $X(d)$.

Proof. This can be proven by means of a refinement of the algorithm to construct a semi-stable model in [KKMS73]. A detailed proof for the case where the base is a smooth k-curve is contained in [Wan97]. Since the proof is essentially combinatorial, it generalizes easily to the case where the base is $\mathrm{Spec}\, R$. $\square$

Proposition 4.4. *Let $\mathcal{X}$ be an snc-model of X, with $\mathcal{X}_k = \sum_{i \in I} N_i E_i$, and let d be a positive integer that is divisible by*

$$e(\mathcal{X}) = \mathrm{lcm}\{N_i \mid i \in I\}.$$

We denote by $\mathcal{X}(d)$ the normalization of $\mathcal{X} \times_R R(d)$. If the $\mu_d(k)$-action on $\mathcal{X}(d)_k(k)$ admits a fixed point, then $\iota_{\mathrm{sp}}(\mathcal{X}) = 1$.

Proof. Let y be a closed point on $\mathcal{X}(d)_k$ that is fixed under the action of $\mu_d(k)$, and let J be the unique subset of I such that the image x of y in $\mathcal{X}_k$ lies in E_J^o. A standard computation shows that the fiber $\mathcal{X}(d) \times_{\mathcal{X}} x$ of $\mathcal{X}(d)$ over x consists of N_J points that are transitively permutated by the action of $\mu_d(k)$; see for instance [Nic13, 2.3.2]. Since y is fixed under the $\mu_d(k)$-action, we conclude that $N_J = 1$, so that $\iota_{\mathrm{sp}}(X) = 1$ by Corollary 2.10. $\square$

Proposition 4.5. *If $\iota_{\mathrm{sp}}(X) > 1$, then there exist a positive integer $d > 0$ and a strictly semi-stable monodromic model $\mathcal{Y}$ of X over $R(d)$ such that the $\mu_d(k)$-action on $\mathcal{Y}_k$ has no k-rational fixed point. For any such $\mathcal{Y}$, we have*

$$\sum_{i \geq 0} (-1)^i \mathrm{Tr}\left(\sigma \mid H^i\left(\mathcal{Y}_k, \mathcal{O}_{\mathcal{Y}_k}\right)\right) = 0.$$

Proof. The existence of $\mathcal{Y}$ follows from Propositions 4.3 and 4.4. Thus it suffices to prove the vanishing of the trace of σ on the coherent cohomology of $\mathcal{Y}_k$. We write

$$\mathcal{Y}_k = \sum_{i \in I} N_i E_i$$

and we set $E_J = \cap_{j \in J} E_j$ for every non-empty subset J of I. We endow the closed subsets E_J of $\mathcal{Y}_k$ with their reduced induced structure.

We denote by n the dimension of X. For every integer m in $\{1, \dots, n+1\}$ we set

$$\mathcal{Y}_k^{(m)} = \coprod_{J \subset I, \, |J| = m} E_J.$$

We denote by a_m the obvious morphism from $\mathcal{Y}_k^{(m)}$ to $\mathcal{Y}_k$. The schemes $\mathcal{Y}_k^{(m)}$ are smooth and proper k-varieties, and the $\mu_d(k)$-action on $\mathcal{Y}_k$ induces a $\mu_d(k)$-action on $\mathcal{Y}_k^{(m)}$ without fixed k-points.

The sequence of $\mathcal{O}_{\mathcal{Y}_k}$-modules

$$\mathcal{O}_{\mathcal{Y}_k} \to (a_1)_* \mathcal{O}_{\mathcal{Y}_k^{(1)}} \to (a_2)_* \mathcal{O}_{\mathcal{Y}_k^{(2)}} \to \ldots \to (a_{n+1})_* \mathcal{O}_{\mathcal{Y}_k^{(n+1)}}$$

is exact, and defines a resolution of the structure sheaf $\mathcal{O}_{\mathcal{Y}_k}$ that we can use to compute the trace of σ on the coherent cohomology of $\mathcal{Y}_k$. By the first spectral sequence for hypercohomology, it suffices to show that

$$\sum_{i \geq 0} (-1)^i \operatorname{Tr}\left(\sigma \mid H^i\left(\mathcal{Y}_k^{(m)}, \mathcal{O}_{\mathcal{Y}_k^{(m)}}\right)\right) = 0$$

for every m in $\{1, \ldots, n+1\}$. This follows from the Woods Hole trace formula [SGA5, III.6.12], since $\mathcal{Y}_k^{(m)}$ is smooth and proper over k and σ is a finite order automorphism of $\mathcal{Y}_k^{(m)}$ without fixed points. □

Theorem 4.6. *Assume that*

$$H^i(X, \mathcal{O}_X) = 0$$

for every $i > 0$. Then $\iota_{\mathrm{sp}}(X) = 1$. In particular, $\iota(X) = 1$, that is, X admits a zero-cycle of degree one.

Proof. Let d be a positive integer and let $\mathcal{Y}$ be a monodromic semi-stable model for X over $R(d)$. We denote by

$$f : \mathcal{Y} \to \operatorname{Spec} R(d)$$

the structural morphism. It follows from Hodge theory that the $R(d)$-modules $R^i f_* \mathcal{O}_{\mathcal{Y}}$ are free; see for instance [IKN05, Theorem 7.1] for a statement that applies in our set-up (log-smooth schemes over $R(d)$). But we also know that $H^i(X(d), \mathcal{O}_{X(d)}) = 0$ for all $i > 0$, so that

$$H^i(\mathcal{Y}_k, \mathcal{O}_{\mathcal{Y}_k}) = 0$$

for every $i > 0$, by [Mum74, II.4 Corollary 3]. Moreover, σ acts trivially on

$$H^0(\mathcal{Y}_k, \mathcal{O}_{\mathcal{Y}_k}) = k.$$

Thus, we find that

$$\sum_{i \geq 0} (-1)^i \operatorname{Tr}(\sigma \mid H^i(\mathcal{Y}_k, \mathcal{O}_{\mathcal{Y}_k})) = 1.$$

Proposition 4.5 now implies that $\iota_{\mathrm{sp}}(X) = 1$. □

(4.7) To conclude, we recall that the triviality of the coherent cohomology of X is not a sufficient condition for the existence of a rational point. In [Laf04], Lafon has constructed an example of an Enriques surface over $\mathbb{C}((t))$ without rational points. It remains an important challenge to understand which additional conditions can be imposed to guarantee the existence of a rational point. A natural candidate seems to be the triviality of the geometric fundamental group (which is satisfied for rationally connected varieties but not for Enriques surfaces). Another interesting problem is whether Theorem 4.6 remains valid in positive residue characteristic.

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